Complex Variables

Holomorphic Functions

Triangle Inequality: $|v+w| \le |v| + |w|$

$||v| - |w|| \le |v-w|$

Cauchy Riemann Equations: $f(x+iy) = u(x,y) + iv(x,y)$

If f is holo at $z_0 = x_0 + iy_0$, then

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \qquad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)$$

Theorem: $f: \mathcal{U} \to \mathbb{C}$, u and v continuously diff + satisfy (CR), then on neighbourhood $\mathcal{U}$, f diff at $z_0 \in \mathcal{U}$.

Useful Functions / Formulae.

- $\exp(z) = e^x(\cos y + i \sin y)$
- $\exp(z+w) = \exp(z)\exp(w)$
- $\exp'(z) = \exp(z)$ holo
- $\exp(z + 2\pi i) = \exp(z)$

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \qquad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2} \qquad \sinh(z) = \frac{\exp(z) - \exp(-z)}{2}$$

$\log(z) = \{ w \in \mathbb{C} : \exp(w) = z \}$

$\log(z) = \{ \ln|z| + i\operatorname{Arg}(z) + 2\pi i k : k \in \mathbb{Z} \}$

$\log(zw) = \log(z) + \log(w) \qquad \log(1/z) = -\log(z)$ } holo except for on branch cut

Complex Powers

Principal branch: $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$

$a, z \in \mathbb{C}$: $z^\alpha = \{ \exp(aw) : w \in \log(z) \}$

$z^\alpha = \{ \exp(\alpha \operatorname{Log}(z)) \exp(i\alpha 2\pi k) : k \in \mathbb{Z} \}$

α	z^α
$\mathbb{Z}$	one unique value
p/q	q values (p,q coprime)
Irrat / Nonreal	∞ many values

$z^\alpha z^\beta = z^{\alpha+\beta}$ for principal branch $z^\alpha = \exp(\alpha \operatorname{Log}(z))$.

Important to know:

→ representing complex numbers in cartesian + polar form.
→ what holomorphic means
→ how to show a function is not holomorphic
→ what the logarithm is, how its multivalued, and how to restrict to one value
→ where a branch of the logarithm fails to be holomorphic
→ Formula for complex powers.

Complex Integrals

$f: \mathbb{R} \to \mathbb{C}$ $\quad \int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

f cont, $f = \frac{dF}{dt}$ for diff $F: [a,b] \to \mathbb{C}$, then $\int_a^b f(t)\,dt = F(b) - F(a)$

$$\left| \int_a^b f(t)\,dt \right| \le \int_a^b |f(t)|\,dt$$

Integrating a complex function along a contour:

Γ regular curve from z_0 to z_1, $f: \Gamma \to \mathbb{C}$ continuous. Then

$$\int_\Gamma f(z)\,dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,dt$$

$\ell(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|\,dt$ (length of a contour)

(M-L lemma) $\left| \int_\Gamma f(z)\,dz \right| \le \max_{z \in \Gamma} |f(z)| \ell(\Gamma)$

$$\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz$$

$D \subseteq \mathbb{C}$, Γ closed contour in D, $f: D \to \mathbb{C}$ cont w/ antider. F. Then

$$\int_\Gamma f(z)\,dz = 0$$

Path Independence Lemma: TFAE:

1) f has an antiderivative F on D
2) $\int_\Gamma f(z)\,dz = 0$ for all closed contours Γ in D, and
3) all contour integrals $\int_\Gamma f(z)\,dz$ are independent of path Γ, only endpoints.

Important to know:

→ If $f: \mathcal{U} \to \mathbb{C}$ is holo, then $\int_\Gamma f(z)\,dz = 0$ for any loop $\Gamma \subseteq \mathcal{U}$
→ If $f: \mathcal{U} \to \mathbb{C}$ is holo, then $\int_\Gamma f(z)\,dz$ depends only on the endpoints of contour Γ.

This first point is:

Cauchy Integral Theorem: Γ a loop, and f holomorphic inside and on Γ. Then $\int_\Gamma f(z)\,dz = 0$

Model Example of integrating a function that is not holo at a single point z_0:

Theorem: $z_0 \in \mathbb{C}$, Γ a loop w/ $z_0 \notin \Gamma$. Then

$$\int_\Gamma \frac{1}{z - z_0}\,dz = \begin{cases} 2\pi i & z_0 \in \operatorname{Int}(\Gamma) \\ 0 & \text{otherwise} \end{cases}$$

Generalizing this:

Cauchy's Integral Formula: Γ loop, $z_0 \in \operatorname{Int}(\Gamma)$, f holo in and on Γ. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\,dz$$

Generalized Cauchy Integral formula: same hypotheses as above, then f is infinitely differentiable at z_0 and $\forall n \in \mathbb{N}$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

Series Expansions for holo functions

Dfn: $\{f_n(z)\}$ converges pointwise to $f(z)$ if $\forall z \in \mathbb{C}$, $\varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t $\forall n \geq N$, $|f_n(z) - f(z)| < \varepsilon$

Dfn: $\{f_n(z)\}$ converges uniformly to $f(z)$ if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t $\forall n \geq N$, $|f_n(z) - f(z)| < \varepsilon \quad \forall z \in \mathbb{C}$.

Weierstrass M-test: $S \subseteq \mathbb{C}$, $f_n : S \to \mathbb{C}$, $M_n \geq 0$ s.t $\forall z \in S$, $n \geq n_0$, $|f_n(z)| < M_n$. Suppose $\sum_{j=0}^{\infty} M_j$ converges. Then $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly on S.

If $f_n \to f$ converges uniformly,

- $\int_\Gamma f_n(z)\, dz \longrightarrow \int_\Gamma f\, dz$ uniformly
- f_n holo $\Rightarrow$ f holo (Morera's)

If $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly, $\int_\Gamma \sum_{j=0}^{\infty} f_j(z)\, dz = \sum_{j=0}^{\infty} \int_\Gamma f_j(z)\, dz$

Power Series

Thm: $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$, with r.o.c R. Then f holo on $D_R(z_0)$.

Taylor Series

Dfn: let $z_0 \in \mathbb{C}$, and f holo at z_0. The Taylor series of f centered at z_0 is the power series

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j$$

Theorem: $z_0 \in \mathbb{C}$, $R > 0$, f holo on $D_R(z_0)$. Then Taylor series for f centered at z_0 converges to $f(z)$ $\forall z \in D_R(z_0)$, and converges uniformly on $\overline{D_r}(z_0)$ $\forall\, 0 \leq r < R$.

Common Taylor Expansions at $z = 0$:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \sin(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \qquad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!} \qquad \cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

Prop: f holo on $D_R(z_0)$, then for $z \in D_R(z_0)$,

$$f'(z) = \sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!} (z - z_0)^j$$

Important to know:

→ every holomorphic function $f : U \to \mathbb{C}$ has a unique power series representation

→ To find power series reps of functions, easiest to combine known ones above.

Laurent Series

Dfn: Let $z_0 \in \mathbb{C}$ be an isolated singularity of $f : U \to \mathbb{C}$, and f holo on the punctured disc of radius r centered at z_0, $D_R'(z_0)$. Then f has a Laurent expansion centered at z_0 that is valid on an annulus $\subseteq D_R'(z_0)$:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

Thm: $z_0 \in \mathbb{C}$, $0 \leq r < R < \infty$, f holo on $A_{r,R}(z_0)$. Then f has a Laurent series expansion which converges on $A_{r,R}(z_0)$, uniformly on $\overline{A}_{r_1,R_1}(z_0)$, w/ coefficients given by

$$a_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{j+1}}\, dz$$

Thm: Laurent series are unique.

General Idea:

→ not every function is holomorphic, but we're still interested in exploring them.

→ we can still study functions that have isolated singularities.

→ These functions don't have power series expansions, but Laurent series expansions

→ when finding a L.S.E, don't use the above integral formula for the coeffs, but rather combine known Taylor / Laurent series.

Cauchy Residue Theorem

Thm 5.1.1: $z_0 \in \mathbb{C}$, f holo on $D_R'(z_0)$ for some $R > 0$ with an isolated singularity at z_0, and $\Gamma \subseteq D_R'(z_0)$, $z_0 \in \text{Int}(\Gamma)$. Then

$$\int_\Gamma f(z)\, dz = 2\pi i\, a_{-1}$$

$\text{Res}(f; z_0) = a_{-1}$ ← coeff. of $(z - z_0)^{-1}$ in Laurent series of f at z_0.

Lemma: $z_0 \in \mathbb{C}$, f holo on $D_R'(z_0)$, pole order m at z_0. Then

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^{m-1} f(z) \right]$$

Important to know:

→ we often want to calculate complex integrals that are tricky.

→ It turns out, however, that often the integral can be computed simply by looking at the Laurent series expansion of the function centered on a singularity contained inside the contour.